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Internal Flow Separation due to Hydromagnetic Effects in a Slowly Varying Exponentially Diverging Channel with Slip at the Permeable Boundaries

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Abstract

The hydromagnetic steady flow of a viscous conducting fluid in a slowly varying exponentially diverging symmetrical channel with slip at the permeable boundaries is investigated. The combined effects of externally applied homogeneous magnetic field and wall slip parameter on the development of internal flow separation in the diverging channel are observed. Analytical solutions are found for the flow governing non-linear boundary-value problem using perturbation method together with Pade´ approximation technique based on computer extended series solution. Our computed results reveal that the axial fluid velocity is reduced by both magnetic field intensity and wall slip parameter and the internal flow separation development at moderately large Reynolds number is suppressed by an increase in magnetic field intensity.

Key words: Diverging channel, magnetic field, internal flow separation, wall slip parameter, Pade´ approximants.

Nomenclature

a	characteristic half-width of the channel	x	axial distance
b	function of wall diverging geometry	y	distance measured in the normal section
B_0	electromagnetic induction	*Greek symbols*	
G	Non-dimensional wall shear stress	ν	kinematic viscosity coefficient
H	Hartmann number	∇	vector differential operator
k	wall slip parameter	ψ	stream function
L	channel characteristic length	ω	vorticity function
Q	fluid flux rate across any section of the channel	μ_e	magnetic permeability
Re	Reynolds number	σ_e	conductivity of the fluid
u	axial fluid velocity component	τ_w	shear stress at the boundary of the channel
v	vertical fluid velocity component	ε	small dimensionless parameter that characterizes the slow variation in the cross-section of the diverging channel

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1. Introduction

The study of flow of an electrically conducting viscous fluid through a diverging channel having permeable walls not only possesses a theoretical appeal but also model many biological and engineering problems such as magnetohydrodynamics (MHD) generators, nuclear reactors, industrial metal casting, plasma studies, blood flow problems, etc. In the past few years, several simple flow problems associated with classical hydrodynamics have received new attention within the more general context of MHD. A survey of MHD studies in the technological fields can be found in **Moreau (1990)**.

The theory of flow of convergent-divergent channels has many applications in aerospace, chemical, civil, environmental, mechanical and bio-mechanical engineering and also in understanding the flow of rivers and canals. A numerical investigation of the study of hydromagnetic flows in a slowly varying exponentially diverging channel under the effect of an externally applied homogeneous magnetic field were conducted by **Makinde** and **Mhone (2006)** using perturbation method and Pade´ approximation technique, **Baker (1975)**.

It is well known that, the flow separates at for values of Reynolds number above a rather moderate critical value if the cross-sectional area of a channel increases gradually with axial distance downstream. However the separated flow is not unique. **Borgas** and **Pedley (1990)** and **Makinde (1997, 1999)** showed analytically that this non-uniqueness occurs at large Reynolds number in channels that are sufficient slowly-varying for the flow to be governed by the boundary layer equations, in which there is neither a transverse pressure gradient nor longitudinal viscous diffusion.

In the present study, the steady hydromagnetic flows in a two-dimensional slowly varying exponentially diverging symmetrical channel with slip at the permeable boundaries under the influence of an externally applied homogeneous magnetic field have been investigated. The objective of the study is to determine numerically the combined effects of the externally applied homogeneous magnetic field and wall slip parameter on the development of internal flow separation as the Reynolds number of the flow increases using perturbation method together with Pade´ approximation technique.

2. Mathematical Formulation

Consider the fluid flow where the fluid has small electrical conductivity and the electromagnetic force produced is very small under the effect of an externally applied homogeneous magnetic field. Let the fluid is flowing through a slowly varying exponentially diverging symmetrical channel with slip at the permeable walls as shown in Figure 1. Let u and v be the velocity components in x and y directions respectively and $b(x)$ defines the wall diverging geometrically. Then, the governing equations for the two-dimensional steady flow, in terms of the vorticity (ω) and stream-function (ψ) can be written as

$$\frac{\partial(\omega,\psi)}{\partial(x,y)} = \upsilon \nabla^2 \omega + \frac{\sigma_e B_0^{\,2}}{\rho} \frac{\partial^2 \psi}{\partial y^2}, \qquad \omega = -\nabla^2 \psi \,, \tag{1}$$

with the appropriate boundary conditions

$$\psi = 0\,, \qquad \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{on} \quad y = 0\,, \tag{2}$$

and $\psi = Q\,, \qquad \frac{\partial \psi}{\partial y} = k\,\frac{db}{dx} \quad \text{on} \quad y = b(x)\,.$ (3)

where $Q = \int_0^{b(x)} u\, dy$ is the fluid flux rate across any section of the channel, k is the wall slip parameter, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $B_0 = (\mu_e H_0)$ the electromagnetic induction, μ_e the magnetic permeability, H_0 the intensity of magnetic field, σ_e the conductivity of the fluid, ρ the fluid density and υ is the kinematic viscosity coefficient.

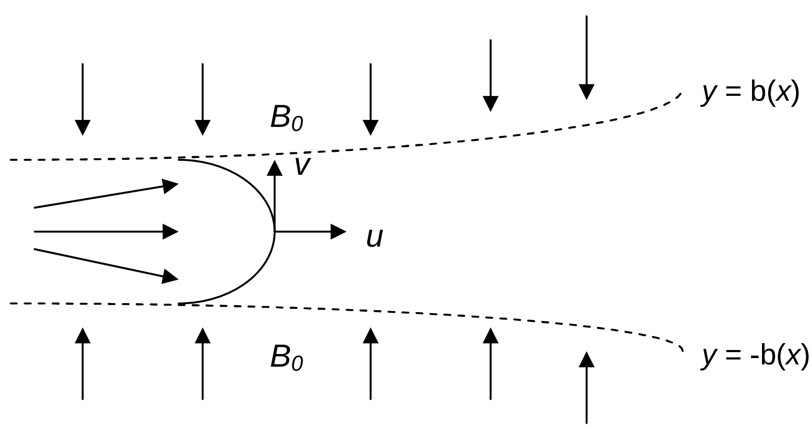

Figure 1.- Geometry of the problem.

The inner surface of the wall is given by $y = \pm b(x/L)$. Let $b(x) = S(\varepsilon x/a)$ where S is the function of x, a is the characteristic half- width of the channel, ε is a small dimensionless parameter that specifies the slow variation in the cross-section of the channel defined as $0 < \varepsilon = a/L << 1$, where L is the channel characteristic length. In the limit $\varepsilon \to 0$, the channel is of constant width. The introduced dimensionless variables are

$$\bar{\omega} = \frac{a^2\,\omega}{Q},\; \bar{x} = \frac{\varepsilon x}{a},\; \bar{y} = \frac{y}{a},\; \bar{\psi} = \frac{\psi}{Q} \quad \text{and} \quad H^2 = \frac{La\sigma_e B_0^{\,2}}{\rho Q}\,. \tag{4}$$

Hence the reduced dimensionless governing equations with the boundary conditions, (neglecting the bars for clarity) can be written as

$$\frac{\partial^2 \omega}{\partial y^2} = \mathrm{Re}\left[\frac{\partial(\omega,\psi)}{\partial(x,y)} - H^2 \frac{\partial^2 \psi}{\partial y^2}\right], \quad \omega = -\frac{\partial^2 \psi}{\partial y^2}, \tag{5}$$

$$\psi = 0, \quad \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{on} \quad y = 0, \tag{6}$$

$$\psi = 1, \quad \frac{\partial \psi}{\partial y} = k\frac{db}{dx} \quad \text{on } y = S. \tag{7}$$

where the flow is considered in the boundary layer approximation or for channel with a small aspect ratio ε, $\mathrm{Re} = \varepsilon Q/\nu$ is the effective flow Reynolds number and H is the magnetic field intensity parameter or Hartmann number. For the geometry of the channel under consideration, S is defined as $S = e^x$.

3. Perturbation Expansion

The equations (5)-(7) are non-linear in nature and therefore it is not possible to find their solutions exactly. However, the solutions can be found in the form of power series in Re i.e.,

$$\psi = \sum_{i=0}^{\infty} \mathrm{Re}^i \psi_i, \qquad \omega = \sum_{i=0}^{\infty} \mathrm{Re}^i \omega_i. \tag{8}$$

Now substitute the expressions in (8) into (5)-(7) and collect the coefficients of like powers of Re. The resulting equations are:

Zeroth Order :

$$\frac{\partial^2 \omega_0}{\partial y^2} = 0, \ \omega_0 = -\frac{\partial^2 \psi_0}{\partial y^2}, \tag{9}$$

$$\psi_0 = 0, \ \frac{\partial^2 \psi_0}{\partial y^2} = 0, \quad \text{on} \ \ y = 0, \tag{10}$$

$$\psi_0 = 1, \ \frac{\partial \psi_0}{\partial y} = k\frac{db}{dx}, \quad \text{on} \quad y = S. \tag{11}$$

Higher Order (n$\geq$1) :

$$\frac{\partial^2 \omega_n}{\partial y^2} = \sum_{i=0}^{n-1} \frac{\partial(\omega_i, \psi_{n-i-1})}{\partial(x,y)} - H^2 \frac{\partial^2 \psi_{n-1}}{\partial y^2}, \quad \omega_n = -\frac{\partial^2 \psi_n}{\partial y^2}, \tag{12}$$

$$\psi_n = 0, \ \frac{\partial^2 \psi_n}{\partial y^2} = 0, \quad \text{on} \quad y = 0, \tag{13}$$

$$\frac{\partial \psi_n}{\partial y} = 0, \ \psi_n = 0, \quad \text{on} \quad y = S. \tag{14}$$

It is difficult to obtain many terms of the solution series manually. So a MAPLE program has been written that calculates successively the coefficients of the solution series. It consists of the following segments:

(i) Declaration of arrays for the solution series coefficients; ψ = array(0……..25), ω = array(0……25).

(ii) Input the leading order term and their derivatives *i.e.* ψ_0, ω_0 .

(iii) Input the modeled channel geometry slope (*i.e.* dS/dx).

(iv) Using a MAPLE loop procedure, iterate to solve equations (12)-(14) for the higher order terms *i.e.* ψ_n, ω_n, n = 1,2,3,…… .

(v) Compute the wall shear stress and the axial pressure gradient.

The first two terms of the solution for stream-function and vorticity are obtained as

$$\psi = [\frac{1}{2}(3 - kS^2)\eta + \frac{1}{2}(kS^2 - 1)\eta^3] + \frac{\text{Re}\, S}{280}[\eta(7H^2S^3k + 5k^2S^4 - 6kS^2 - 7H^2S + 15) - \eta^3(14H^2S^3k + 11k^2S^4 - 16kS^2 - 14H^2S + 33) + 7\eta^5(H^2S^3k + k^2S^4 - 2kS^2 - H^2S + 3) - \eta^7(k^2S^4 - 4kS^2 + 3)] + ... \tag{15}$$

$$\omega = [\frac{3\eta}{S^2}(1 - kS^2)] + \frac{3\text{Re}}{140S}[\eta(14H^2S^3k + 11k^2S^4 - 16kS^2 - 14H^2S + 33) - \frac{70\eta^3}{3}(H^2S^3k + k^2S^4 - 2kS^2 - H^2S + 3) + 7\eta^5(k^2S^4 - 4kS^2 + 3)] + ... \tag{16}$$

where $\eta = y/S$. The shear stress at the boundary of the channel is given by

$$\tau_w = -\frac{1}{1+b_x^{\,2}}\left[\left(\sigma_{yy} - \sigma_{xx}\right)b_x + \left(1 - b_x^{\,2}\right)\sigma_{xy}\right] \quad \text{on } y = b(x) \tag{17}$$

where σ_{yy} , σ_{xx} , σ_{xy} are the usual stress components, *i.e.*,

$$\begin{aligned} \sigma_{xy} &= \mu\left[\psi_{yy} - \psi_{xx}\right] \\ \sigma_{yy} - \sigma_{xx} &= -4\mu\psi_{xy} \end{aligned} \tag{18}$$

The subscripts (x, y) denote partial differentiation with respect to (x, y), respectively. The dimensionless form of wall shear stress can be written as:

$$G = \frac{a^2 S^2}{\mu Q}\tau_w = -\frac{S^2}{\left(1+\varepsilon^2 S_x^{\,2}\right)}\left[\left(\psi_{yy} - \varepsilon^2\psi_{xx}\right)\left(1 - \varepsilon^2 S_x^{\,2}\right) - 4\varepsilon^2 S_x \psi_{xy}\right] \quad \text{on } y = S \tag{19}$$

and for $0 < \varepsilon \ll 1$ we obtain

$$G = \left(3 - 3S^2 k\right)\mathrm{Re}^0 + S\left(-0.34286 + 0.2\,SH^2 + 0.05714\,S^2 k - 0.2\,S^3 H^2 k - 0.11428\,S^4 k^2\right)\mathrm{Re} + \ldots \tag{20}$$

4. Internal Flow Separation

We have investigated the solution behavior by algebraic programming language (MAPLE). The first 19 coefficients for the above solution series have been obtained which represent the flow characteristics. The above series are reformed into several diagonal Pade´ approximants of order $N = M + M$ as

$$G = \sum_{i=0}^{N} f_i\,\mathrm{Re}^i = \frac{\sum_{i=0}^{M} a_i\,\mathrm{Re}^i}{\sum_{i=0}^{M} c_i\,\mathrm{Re}^i}. \tag{21}$$

This method fails when the denominator of the fraction is evaluated near the zeros. By equating the numerator of equation (21) to zero we have computed the Reynolds number at which separation occurs in the flow field (*i.e.* $G \to 0$) for different values of k and H at position S = 1on the channel, as shown in Table 1.

Table 1

Computations showing the Reynolds number for internal flow separation development in the diverging channel at S = 1.

$k\downarrow$	$H\rightarrow$	0.0	0.5	1.0	1.5
0.1	Re	6.93516	7.98599	12.56802	22.62489
0.01	Re	7.52575	9.12779	16.59612	29.58886
0.001	Re	7.42219	9.23728	16.96519	29.59017
0.0	Re	7.93237	9.26567	17.08932	32.97350

5. Results and Discussion

The combined effects of homogeneous magnetic field and wall slip parameter have been investigated. Flow separations have been observed in the slowly varying exponentially diverging channel with slip at the permeable boundaries due to hydromagnetic effects and computed numerically as shown in Table 1.

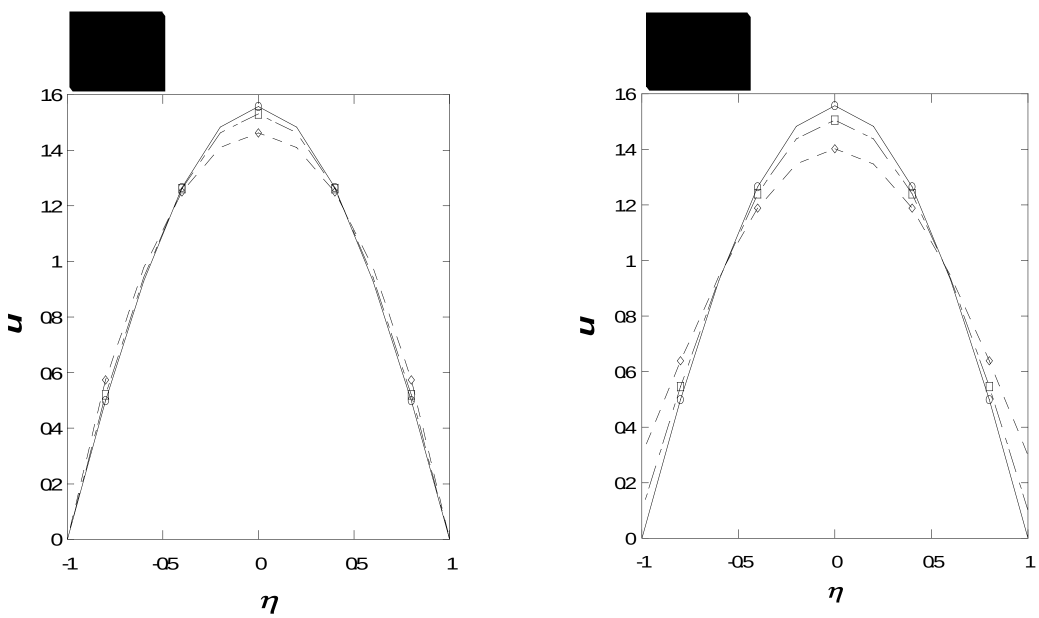

Figure 2: Axial velocity profiles for different values of *H* and *k* ; *S* = 1 and Re = 1.

Figure 2 show the axial velocity profiles at the given position in the diverging channel. A parabolic axial velocity profile is observed with maximum value at the channel centerline and minimum value at the walls. It is also clear that around the centerline of the channel the axial fluid velocity is reduced with an increase in both magnetic field intensity (*H*) and wall slip parameter (*k*).

Figures 3(a)-3(b) represent the wall shear stress (G) with respect to flow Reynolds number at $S = 1$ in the diverging channel. Here it is interesting to note that the requirement of flow Reynolds number for the development of internal flow separation in the slowly varying exponentially diverging channel increases as the magnetic field intensity increases in magnitude, which is also clear from Table 1. Meanwhile, a further increase in magnetic field intensity may suppress or totally prevent the development of internal flow separation in the diverging channel. Generally, early separation is observed with an increase in wall slip parameter. From Table 1 and Figures 3(a)-3(b), it is also clear that if flow Reynolds number is sufficiently high, internal flow separation development is still possible at low magnetic field intensity. Hence, in order to prevent the occurrence of internal flow separation in the diverging channel, the imposed external magnetic field intensity on the conducting fluid must be sufficiently high, and this prevention will be accelerated for the lower value of wall slip parameter which is clear from Table 1. So, both magnetic field and wall slip parameter have great influence on the development of internal flow separation in the diverging channel.

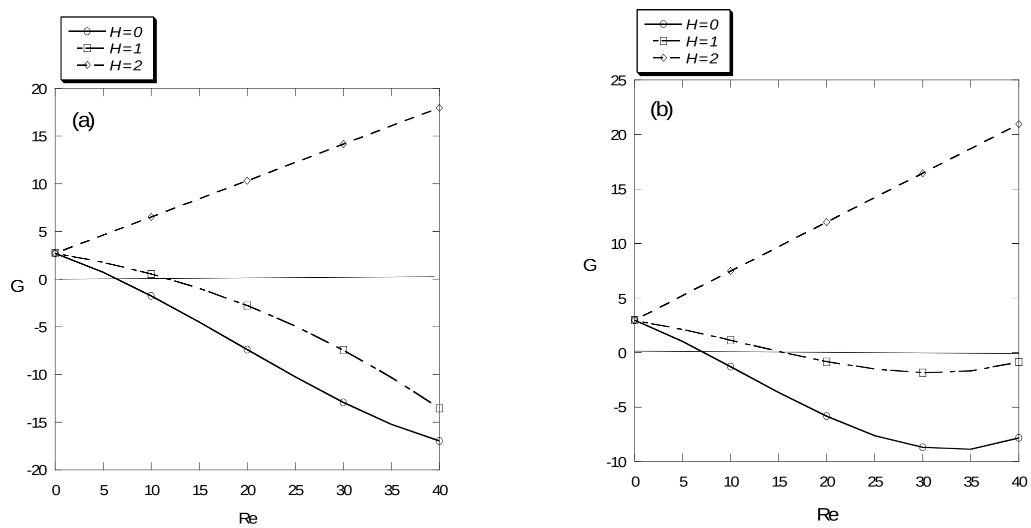

Figure 3: Wall shear stress for different values of H; S = 1; k = (a) 0.1 and (b) 0.01.

Conclusion

We investigated the effects of the externally applied homogeneous magnetic field intensity and permeable parameter simultaneously on the flow for the development of internal flow separation at a given position in the diverging channel. Our computed results revealed that the axial fluid velocity is reduced by both magnetic field intensity and wall slip parameter and for internal flow separation development the requirement of flow Reynolds number increases with an increase in magnetic field intensity and decrease in wall slip parameter. The number of coefficients for the solution series above 19 would give us more accurate values of the flow Reynolds number for the internal flow separation development. Again the approximant methods such as high order differential approximants, Drazin-Tourigny approximants, etc. may produce better results.

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