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On The Generalized Gaussian and Mean curvatures in E_1^{n+1}

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Abstract:
Before now in [2] the generalized Gaussian and mean curvatures were proved in Euclidean space, but now we prove the theorems in Lorentzian Space. In our previous paper, we have studied higher order Gaussian curvatures in Lorentzian space. This allowed us to prove that

$$\varphi(P)=\prod_{i=1}^{n}(1+\varepsilon_i r k_i)$$

In addition to Gaussian and mean curvatures, K_r and H_r for parallel surfaces in E_1^3 are given. In this study by means of higher order Gaussian and mean curvatures we calculate the generalized curvatures K_r and H_r for parallel surfaces in E_1^{n+1}.

Keywords: Gaussian curvatures, mean curvatures, parallel hypersurfaces, higher order Gaussian curvatures.

1. Introduction

For an integer v with $0 \leq v \leq n$, changing the first v plus signs above to minus gives a metric tensor

$$< v_p, w_p >= -\sum_{i=1}^{v} v_i w_i + \sum_{j=v+1}^{n} v_i w_i$$

of index v .The resulting semi-Euclidean space R_v^n reduces to R^n if $v = 0$. For $n \geq 2$, R_1^n is called Minkowski $n-space$;if n= 4 it is the simplest example of a relativistic spacetime.
The common value v of index $g_p$on a semi-Riemannian manifold $M$is a called the index of $0 \leq v \leq n = dimM$. If $v = 0$, $M$is $Riemannian\ manifold$; each $g_p$is then a (positive definite) inner product on $T_p(M)$. If $v = 1\ and\ n \geq 2$, M is a Lorentz manifold.
Fix the notation

$$\varepsilon_i = \begin{cases} -1 & for\ 1 \leq i \leq v, \\ +1 & for\ v+1 \leq i \leq n \end{cases}.$$

Then the metric of R_v^n can be written

$$g = \sum \varepsilon_i du_i \otimes du_i.$$

A tangent vector v to M is

$$spacelike\ if\ < v, v >> 0\ or\ < v, v >= 0,$$
$$null\ if\ < v, v >= 0\ and\ v \neq 0$$
$$timelike\ if\ < v, v >< 0 \quad [1]$$

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Let M and $\bar{M}$ are two hypersurfaces in E_1^{n+1} with unit normal vector $N$of M.

$$N = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial_{x_i}}$$

where each α_i is a C^∞function of M. If there is a function f, from M to $\bar{M}$ such that

$$f: M \to \bar{M}$$
$$P \to f(P) = P + rN_P$$

then $\bar{M}$ is called parallel hypersurfaces of M, where $r \in R$.[1]

S denotes the shape operator on M, at $P \in M$.The function H defined by

$$H: M \to R$$

$$P \to H(P) = Trace\, S(P)$$

is called the mean curvature function of M and the real number $H(P)$ is called mean curvature of M at the point P. [3]

The function K defined by

$$K: M \to R$$

$$P \to K(P) = \varepsilon det\, S(P)$$

is called the Gaussian curvature function of M and the real number $K(P)$ is called Gaussian curvature of M at the point P. [3]

Definition :

Let M be a hypersurfaces in $E_1^{n+1}$and $T_M(P)$ be a tangent space on M, at P∈M. If S_P denotes the shape operator on M, then

$$S_P: T_M(P) \to T_M(P)$$

is a linear mapping. If we denote the characteristic vectors by $k_1, k_2, \ldots, k_n$ and the corresponding characteristic vectors by $x_1, x_2, \ldots, x_n$ of S_P then $k_1, k_2, \ldots, k_n$ are the principal curvatures and $x_1, x_2, \ldots, x_n$ are the principal directions of M, at P∈M . On the other hand, if we use the notions ε_i=+1

$$K_1^{(n)}(k_1, k_2, \ldots, k_n) = \varepsilon_1 k_1 + \sum_{i=2}^{n} \varepsilon_i k_i$$

$$K_2^{(n)}(k_1, k_2, \ldots, k_n) = \sum_{i=1<j}^{n} \varepsilon_i k_i k_j + \sum_{i\neq 1<j}^{n} \varepsilon_i k_i k_j$$

$$K_3^{(n)}(k_1, k_2, \ldots, k_n) = \sum_{i=1<j<t}^{n} \varepsilon_i k_i k_j k_t + \sum_{i\neq 1<j<t}^{n} \varepsilon_i k_i k_j k_t$$

$$\vdots$$

$$K_n^{(n)}(k^1, k^2, \ldots, k_n) = \varepsilon_1 \prod_{i=1}^{n} k_i$$

then the characteristic polynomial of S(P) becomes

$$P_{S(P)}(k) = k^{n} + (-1)K_1^{(n)}k^{n-1} + \ldots + (-1)^{n}K_n^{(n)}$$

and $K_1, K_2, \ldots, K_n$ are uniquely determined , where the functions K_i are called the higher ordered Gaussian curvatures of the hypersurface M.

Theorem 1:

Let M be a hypersurfaces in E_1^{n+1} and $K_1, K_2, \ldots, K_n$ are called the higher order Gaussian curvatures and $k_1, k_2, \ldots, k_n$ are the principal curvatures at the point $f(P) \in M$. Let us define a function ε_i =+1 and $\varepsilon_1 = \pm 1$ $(i \neq +1)$

$$\varphi: M \to R$$

$$P \to \varphi(P) = \varphi(r, k_1, k_2, \ldots, k_n)$$
$$= \prod_{i=1}^{n} (1 + \varepsilon_i r k_i)$$

such that φ function is

$$\varphi(r, k_1, k_2, \ldots, k_n) = 1 + rK_1 + r^2 K_2 + \cdots + r^n K_n .$$

$1 \leq i \leq n$.

Proof:

We prove the theorem by induction method.

a) If X_p is spacelike ε_1=+1, for $n = 1$, the theorem holds. Actually,

$$\varphi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{1} 1 + \varepsilon_i r k_i$$

$$= 1 + r k_1$$

$$= 1 + r \sum_{i=1}^{1} \varepsilon_i k_i$$

$$= 1 + rK_1$$

Now suppose that the theorem holds for $n-1$ and show that is true for n:

$$\varphi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n-1} (1 + \varepsilon_i r k_i)$$

$$= 1 + r \sum_{i=1}^{n-1} k_i + r^2 \sum_{i<j}^{n-1} k_i k_j + \cdots + r^{n-1} \prod_{i=1}^{n-1} k_i$$

$$= 1 + rK_1 + r^2 K_2 + \cdots + r^{n-1} K_{n-1}$$

For n, both sides of the equation is multiplied by $(1 + rk_n)$

$$\left(\prod_{i=1}^{n-1} 1 + rk_i\right)(1 + rk_n) = \left(1 + r\sum_{i=1}^{n-1} k_i + r^2 \sum_{i<j}^{n-1} k_i\, k_j + \cdots + r^{n-1}\prod_{i=1}^{n-1} k_i\right)(1 + rk_n)$$

$$= 1 + r\left(\sum_{i=1}^{n-1} k_i + k_n\right) + r^2\left(\sum_{i<j}^{n-1} k_i\, k_j + k_n \sum_{i=1}^{n-1} k_i\right) + \cdots + r^n k_n \prod_{i=1}^{n-1} k_i$$

and we have

$$\prod_{i=1}^{n} 1 + rk_i = 1 + r\sum_{i=1}^{n} k_i + r^2 \sum_{i<j}^{n} k_i\, k_j + \cdots + r^n \prod_{i=1}^{n} k_i$$

$$\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n) = 1 + \ \mathrm{r}K_1 + r^2 K_2 + \cdots + r^n K_n$$

b) If X_p is timelike ε_1=-1, for $n = 1$, the theorem holds . Actually,

$$\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n) = \prod_{i=1}^{1} 1 + \varepsilon_i r k_i$$

$$= 1 + \varepsilon_1 r k_1$$

$$= 1 + r\sum_{\mathrm{i}=1}^{1} \varepsilon_{\mathrm{i}}\, k_i$$

$$= 1 + rK_1$$

Now suppose that the theorem holds for $n - 1$ and show that is true for n:

$$\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n) = \prod_{i=1}^{n-1} 1 + \varepsilon_i r k_i$$

$$= 1 + r\sum_{i=1}^{n-1} \varepsilon_i k_i + r^2 \sum_{i<j}^{n-1} \varepsilon_i k_i k_j + \cdots + r^{n-1} \prod_{i=1}^{n-1} \varepsilon_i\ \ k_i$$

$$= 1 + \mathrm{r}K_1 + r^2\ K_2 + \cdots + r^{n-1} K_{n-1}$$

For n, both sides of the equation is multiplied by $1 + r\varepsilon_n k_n$

$$\left(\prod_{i=1}^{n-1} 1 + r\varepsilon_i\, k_i\right)(1 + r\varepsilon_n k_n) = \left(1 + r\sum_{i=1}^{n-1} \varepsilon_i\, k_i + r^2 \sum_{i<j}^{n-1} \varepsilon_i k_i\, k_j + \cdots + r^{n-1}\prod_{i=1}^{n-1} \varepsilon_i\, k_i\right)(1 + r\varepsilon_n k_n)$$

$$= 1 + r\left(\sum_{i=1}^{n-1} \varepsilon_i k_i + \varepsilon_n k_n\right) + r^2\left(\sum_{i<j}^{n-1} \varepsilon_i k_i\, k_j + k_n \sum_{i=1}^{n-1} \varepsilon_i\, k_i\right) + \cdots + r^n \varepsilon_n\, k_n \prod_{i=1}^{n-1} \varepsilon_i\, k_i$$

and we have

$$\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n) = 1 + \mathrm{r}K_1 + r^2K_2 + \cdots + r^nK_n$$

Theorem 2:
Let M be a hypersurfaces in E_1^{n+1} and $K_1, K_2, \ldots, K_n$ are called the higher order Gaussian curvatures and $\mathrm{k}_1, \mathrm{k}_2, \ldots, k_n$ are the principal curvatures at the point $f(P) \in M$. $\bar{K}$ and $\bar{H}$ are generalized Gaussian and mean curvatures of $\bar{M}$ at the point $f(P)$. Suppose the function

$$\varphi: M \to R$$

$$P \to \varphi(P) = \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n) = \prod_{i=1}^{n}(1 + \varepsilon_i r k_i)$$

such that ε_i =+1 and ε_1=±1 (i≠+1)
Then we have

$$\bar{K} = \varepsilon \frac{\frac{\partial^n \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n)}{\partial r^n}}{n!\, \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n)}$$

and

$$\bar{H} = \frac{1}{n} \frac{\frac{\partial \varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n)}{\partial r}}{\varphi(\mathrm{r}, \mathrm{k}_1, \mathrm{k}_2, \ldots, k_n)}$$

Proof: If k is principal curvatures of M at the point P in direction X, then $\frac{k}{1+rk}$ is the principal curvatures of $\overline{M}$ at the point $f(P)$ in direction $f_*(X)$, that is, $\bar{S}(f_*(X)) = \frac{k_1}{1+rk_1} f_*(X)$ which means that f preserves principal directions, where f_* is the differential of f and we know that [3]

$$\bar{S}(f_*(X_1)) = \frac{k_1}{1+rk_1} f_*(X_1)$$

$$\bar{S}(f_*(X_2)) = \frac{k_2}{1+rk_2} f_*(X_2)$$

$$\vdots$$

$$\bar{S}(f_*(X_n)) = \frac{k_n}{1+rk_n} f_*(X_n).$$

then we know that the shape operator of $\overline{M}$ is

$$S_r = \begin{bmatrix} \frac{\varepsilon_1 k_1}{1 + r\varepsilon_1 k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\varepsilon_n k_n}{1 + r\varepsilon_n k_n} \end{bmatrix}$$

and

$$\bar{K} = \det S_r$$

$$=\varepsilon\left(\frac{\varepsilon_1 k_1}{1+r\varepsilon_1 k_1}\dots\frac{\varepsilon_n k_n}{1+r\varepsilon_n k_n}\right)$$

$$=\varepsilon\frac{\varepsilon_1 k_1 \varepsilon_2 k_2 \dots \varepsilon_n k_n}{\prod_{i=1}^{n}(1+\varepsilon_i r k_i)}$$

$$=\varepsilon\frac{\prod_{i=1}^{n}\varepsilon_i k_i}{\prod_{i=1}^{n}(1+\varepsilon_i r k_i)}$$

We multiply the right sides of the equation with $n!$

$$=\varepsilon\frac{n!\prod_{i=1}^{n}\varepsilon_i k_i}{n!\prod_{i=1}^{n}(1+\varepsilon_i r k_i)}$$

$$=\varepsilon\frac{n!\,K_n}{n!\,\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\dots,k_n)}$$

and we derivate to $\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\dots,k_n)$ order n according to r

$$\frac{\partial\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\dots,k_n)}{\partial r}=\frac{\partial(1+\mathrm{r}K_1+r^2K_2+\cdots+r^nK_n)}{\partial r}$$

$$=K_1+2rK_2+\cdots+nr^{n-1}K_n$$

$$\frac{\partial^2\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\dots,k_n)}{\partial r^2}=\frac{\partial(K_1+2rK_2+\cdots+nr^{n-1}K_n)}{\partial r}$$

$$=2K_2+\cdots+n(n-1)r^{n-2}K_n$$

and we continue to derivation, we have

$$\frac{\partial^n\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\dots,k_n)}{\partial r^n}=n!\,K_n$$

and we obtain with implying equality

$$\bar{K}=\varepsilon\frac{\frac{\partial^n\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\dots,k_n)}{\partial r^n}}{n!\,\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\dots,k_n)}.$$

We proof the other equality

$$\bar{H}=\frac{1}{n}Iz\,S_r$$

$$=\frac{1}{n}\left(\frac{\varepsilon_1 k_1}{1+r\varepsilon_1 k_1}+\cdots+\frac{\varepsilon_n k_n}{1+r\varepsilon_n k_n}\right)$$

$$=\frac{1}{n}\left(\frac{\varepsilon_1 k_1\prod_{i=2}^{n}(1+\varepsilon_i r k_i)+\varepsilon_2 k_2\,(1+\varepsilon_1 r k_1)\prod_{i=3}^{n}(1+\varepsilon_i r k_i)+\cdots+\varepsilon_n k_n\,\prod_{i=1}^{n-1}(\varepsilon_i r k_i)}{\prod_{i=1}^{n}(1+\varepsilon_i r k_i)}\right)$$

We derivate according to r

$$\frac{\partial \varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\ldots,k_n)}{\partial r}=\frac{\partial(\prod_{i=1}^{n}(1+\varepsilon_i r k_i))}{\partial r}$$

$$=\frac{\partial\big((1+\varepsilon_1 r k_1)(1+\varepsilon_2 r k_2)\ldots(1+\varepsilon_n r k_n)\big)}{\partial r}$$

$$=\varepsilon_1 k_1(1+\varepsilon_2 r k_2)(1+\varepsilon_3 r k_3)\ \ldots(1+\varepsilon_n r k_n)$$

$$+(1+\varepsilon_1 r k_1)\varepsilon_2 k_2(1+\varepsilon_3 r k_3)\ldots(1+\varepsilon_n r k_n)$$

$$+(1+\varepsilon_1 r k_1)(1+\varepsilon_2 r k_2)\varepsilon_3 k_3\ldots(1+\varepsilon_n r k_n)$$

$$\cdots$$

$$+(1+\varepsilon_1 r k_1)(1+\varepsilon_2 r k_2)(1+\varepsilon_3 r k_3)\ldots(1+\varepsilon_{n-1} r k_{n-1})\varepsilon_n k_n$$

$$=\varepsilon_1 k_1\prod_{i=2}^{n}(1+\varepsilon_i r k_i)+\varepsilon_2 k_2\,(1+\varepsilon_1 r k_1)\prod_{i=3}^{n}(1+\varepsilon_i r k_i)+\cdots+\varepsilon_n k_n\prod_{i=1}^{n-1}(\varepsilon_i r k_i)$$

So we have last equation and we obtain that

$$\overline{H}=\frac{1}{n}\frac{\frac{\partial \varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\ldots,k_n)}{\partial r}}{\varphi(\mathrm{r},\mathrm{k}_1,\mathrm{k}_2,\ldots,k_n)}.$$

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