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STATISTICAL ANALYSIS OF MAXIMUM (MINIMUM) SCORES IN 'C' MATCHED SAMPLES

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ABSTRACT

This paper proposes a statistical method for the analysis of maximum (or minimum) scores or performances by subjects in a series of tests and for testing the hypothesis that subjects are on the average equally likely to achieve, perform or earn highest(best, largest) scores or grades under a set of judges, treatment conditions, time periods or locations. The sampled populations may be measurements on as low as the ordinal scale and need not be continuous or numeric. Necessary test statistic is developed that may also be used to determine which of the treatment levels may account for any possible rejection of the initial null hypothesis. The proposed method is illustrated with some sample data and shown to be at least as powerful as the Cochran Q test.

KEYWORDS: C-matched, Degree of freedom, Chi-square test, Cochran Q test, treatment, maximum scores, minimum scores.

INTRODUCTION

If a researcher has collected random samples of observations from some populations that are continuous, homogenous and normally distributed, the researcher could use the usual parametric methods to test any desired hypothesis concerning the estimates of the population mean (Oyeka,2009). However if these populations do not satisfy the usual assumptions for the valid use of parametric test or if the null hypothesis concerns the modes of the population, then parametric tests may not be validly used to test the null hypothesis (Gibbons,1971,Oyeka,2009). This is because the mode of a population distribution unlike the mean often has intractable distributional properties that are rather difficult to evaluate in practical applications (Gibbons,1971). These include situations in which one has sample data on economic activities, say, transactions in the stock exchange overtime or space or in education or job interview when one has data on the performance of students or candidate over time or space or in the evaluation of the health status of subjects over time or space; etc. In each of these and similar situations research interest may be in statistically determining the maximum, highest or peak score, or the lowest, smallest or troughs in the set of scores by subjects over time or space and to statistically determine whether there is any significant difference between the treatment levels represented in terms of time or space in their maximum and minimum scores (Spiegel,1998;Oyek et al,2012). We therefore here propose to develop a nonparametric statistical method for the comparison of modes or troughs of populations matched in time or space that do not require any distribution assumptions.

THE PROPOSED METHOD

Let x_{ij} be the i^{th} sample, block or batch of observations randomly drawn from population x_j, for $i = 1,2,\ldots,n;\ j = 1,2,\ldots,c\ (c \geq 2)$. It is assumed that the c-sampled populations or treatment levels are related either in time or space and may be measurements on as low as the ordinal scale (Gibbons, 1971). They also need not be continuous. To develop a test statistic that the maximum (minimum) score or observation is as likely to occur in any one treatment level as in another, we let

$$u_{ij} = \begin{cases} 1, \text{ if } x_{ij} \text{ is the highest (best, largest) score in sample, row or block } i \text{ and treatment level } j \\ 0, \text{ otherwise.} \end{cases} \quad \ldots\ldots(1)$$

For $i = 1,2,\ldots n,\ j = 1,2,\ldots,c$

Note that Equ 1 may also be used to test similar null hypothesis about minimum scores if x_{ij} is redefined as the lowest (worst, smaller) score in the ith batch or block or by the ith subject at the jth treatment level, time period or location for i=1,2..,n;j=1,2,..c. Note also that $u_{ij} = 1$,for all treatment levels 'j' in which the maximum score or observation occurs for each subject 'I'.i=1,2,…,n;j=1,2,…,c. Cochran Q test may be used to test the null hypothesis by first coding all maximum (or minimum) scores 1 in each treatment level j and other observations 0 for the 'I' block or subject,i=1,2,..,n;j=1,2,…,c;that is by using the result of Equ 1(Gibbons, 1973;Oyeka,2009).We will however here propose and develop an alternative approach for the same purpose.

Let

$$®_j = p(u_{ij} = 1) \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots(2)$$

Define

$$W_j = \sum_{i=1}^{n} u_{ij} \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ ..(3)$$

And

$$W = \sum_{j=1}^{c} W_j = \sum_{j=1}^{c}\sum_{i=1}^{n} u_{ij} \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ (4)$$

A null hypothesis that is usually of general interest is that each of the c treatment levels is on the average equally as likely to contain the highest (best, largest) score or observation as any other treatment level for all blocks. In other words, the null hypothesis of interest would be:

$$H_0 : ®_2 = \ldots = ®_c = ® \ \textit{versus}$$

$$H_1 : ®_j \neq ®_l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$j,l = 1,2,\ldots c;\ j \neq l$$

$$E(u_{ij}) = ®_j;\ \operatorname{var}(u_{ij}) = ®_j(1 - ®_j) \qquad \ldots\ldots\ \ldots\ldots\ \ldots(6)$$

Equation 6 shows the expected value and variance of W_j respectively.

$$E(W_j) = \sum_{i=1}^{n} E(u_{ij}) = n®_j;\ Var(W_j) = \sum_{i=1}^{n} \operatorname{var}(u_{ij}) = n.®_j(1 - ®_j) \ldots\ldots(7)$$

Also the expected value and variance of W are respectively

$$E(W) = \sum_{j=1}^{c} E(W_j) = n\sum_{j=1}^{c} ®_j;\ Var(W) = \sum_{j=1}^{c} \operatorname{var}(W_j) = n.\sum_{j=1}^{c} ®_j(1 - ®_j) \ldots\ldots\ldots(8)$$

A test statistic for the null hypothesis of Equ 5 could be developed based on W of Eqn 4 by finding the sampling distribution of W using Equ 1-4 and 6-8.This procedure is however rather tedious and cumbersome. We will here adopt an alternative approach based on the chi-square test for independence. Now note that $®_j$ is the probability that on the average the highest (best, largest) observation or score occurs at the jth treatment level, time period or location for all rows or blocks of subjects for j=1,2,…,c. Its sample estimate is

$$p_j = \hat{®}_j = \frac{w_j}{n} = \frac{f_j}{n} \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots(9)$$

Where f_j is the total number of 1s in u_{ij}, that is the total number of times the highest (best, largest) observation or score occurs at the jth treatment level, time period or location; for j=1,2,…c, for all i=1,2,…n, that is for all subjects or blocks of subjects. Now the overall or total number of 1s, that is the total number of highest (best, largest) scores or observations for all treatment levels, time periods or locations is

$$f = \sum_{j=1}^{c}\sum_{i=1}^{n} u_{ij} = \sum_{j=1}^{c} f_j \qquad 10$$

To develop a test statistic for the null hypothesis of Equ 5 based on the chi-square test for independence; we note that the total number of 1s and 0s, that is the total number of times the highest (best, largest) observations or scores occur and do not occur at the treatment levels, time period or location for all subjects or blocks that is observed number of 1s and 0s are

$$o_{1j} = f_j;\ o_{2j} = n - f_j \ \ for\ j = 1,2,\ldots,c. \qquad 11$$

The corresponding sample proportions are

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$$p_j = \pi_j = \frac{f_j}{n}; q_j = \frac{n-f_j}{n} = 1-p_j \qquad 12$$

The overall or total number of 1s and 0s are respectively

$$n_1 = f = \sum_{j=1}^{c} f_j ; n_2 = \sum_{j=1}^{c}(n-f_j) = n.c - f \qquad 13$$

Now under the null hypothesis of Equ 5 the expected number of 1s and 0s ,that is the expected number of times the highest (best, largest) observations or scores occur and do not occur at the jth treatment level, time period or location for all subjects or blocks of subjects are respectively

$$E_{1j} = \frac{n.f}{n.c} = \frac{f}{c}; E_{2j} = \frac{n(n.c-f)}{n.c} = \frac{n.c-f}{c} \qquad 14$$

The corresponding sample proportions are respectively

$$\bar{P} = \frac{f}{n.c}; \bar{q} = \frac{n.c-f}{n.c} = 1-\bar{p} \qquad 15$$

Hence if the null hypothesis of Equ 5 is true then the corresponding Chi-square test statistic is

$$\chi^2 = \sum_{i=1}^{2}\sum_{j=1}^{c} \frac{(0_{ij}-E_{ij})^2}{E_{ij}} \qquad 16$$

Which under H_0 has approximately the chi-square distribution with (2-1)(c-1)=(c-1) degrees of freedom for sufficiently large 'n'.

Now substituting Equs 11 and 14 in Equ 16 gives

$$\chi^2 = \frac{\sum_{j=1}^{c}\left(f_j - \frac{f}{c}\right)^2}{\frac{f}{c}} + \frac{\sum_{j=1}^{c}\left((n-f_j)-(n.c-f)\right)^2}{\frac{n.c-f}{c}}$$

Which when further simplified becomes

$$\chi^2 = \frac{n.c^2 \sum_{j=1}^{c}\left(f_j - \frac{f}{c}\right)^2}{f(n.c-f)} \qquad 17$$

Or equivalently in terms of the sample proportions of Equs 12 and 15

$$\chi^2 = \frac{n.\sum_{j=1}^{c}\left(p_j - \bar{p}\right)^2}{\bar{p}\bar{q}} = \frac{n.\left(\sum_{j=1}^{c} p_j^2 - c.\bar{p}^2\right)}{\bar{p}\bar{q}} \qquad 18$$

Which under H_0 has approximately the chi-square distribution with c-1 degrees of freedom for sufficiently large n. The null hypothesis H_0 of Equ 5 is rejected at the α level of significance if

$$\chi^2 \le \chi^2_{1-\alpha;c-1} \qquad 19$$

Otherwise H_0 is accepted.

If the null hypothesis of Equ 5 is rejected one may proceed further to determine which treatment level, time period or location, or their combinations may have led to the rejection of H_0.This may be achieved by partitioning the chi-square test statistic of Equ 17 or Equ 18 in the usual way, that is by temporarily omitting or dropping the treatment level that has the largest combination to the overall chi-square value and repeating the analysis with the remaining treatment levels. This process is continued until no significant difference is found to exist between the remaining treatment levels, at each stage performing necessary comparisons in the usual way.

ILLUSTRATIVE EXAMPLE

A random sample of 15 candidates attending a job interview were assessed by a panel of 5 judges on a 11 point scale from 0(worst) to best (10) with the following results (Table 1).

Table 1: Scores by candidates in a job interview under 5 judges.

Candidate No.	Judge 1	Judge 2	Judge 3	Judge 4	Judge 5	Highest score
1	3	4	6	9	3	9
2	3	1	10	9	8	10
3	3	2	10	9	8	10
4	6	4	9	1	9	9
5	3	7	8	2	8	8
6	5	5	3	4	2	5
7	1	4	9	3	8	9
8	7	3	2	7	5	7
9	8	2	6	4	5	8
10	3	8	2	2	8	8
11	2	2	3	3	7	7
12	10	3	5	9	4	10
13	5	6	7	7	10	10
14	9	5	10	3	7	10
15	2	6	9	5	8	9

To analyze the data of Tble 1 using the proposed method we apply Equ 1 to the values of the scores in the Table for each candidate to obtain the values of u_{ij} consistent with the highest values of the scores shown in the last column of Table 1 for each study. The results are shown in Table 2.

Table 2: Values of u_{ij} (Equ 1) for the data of Table 1

Candidate No.	Judge 1	Judge 2	Judge 3	Judge 4	Judge 5	Total(B_i)
1	0	0	0	1	0	1
2	0	0	1	0	0	1
3	0	0	1	0	0	1
4	0	0	1	0	1	2
5	0	0	1	0	1	2
6	1	1	0	0	0	2
7	0	0	1	0	0	1
8	1	0	0	1	0	2
9	1	0	0	0	0	1
10	0	1	0	0	1	2
11	0	0	0	0	1	1
12	1	0	0	0	0	1
13	0	0	0	0	1	1
14	0	0	1	0	0	1
15	0	0	1	0	0	1

Total(n)	15	15	15	15	15	75(n.c)
No of 1s (f_j)	4	2	7	2	5	20(f)
No of 0s (f_j)	11	13	8	13	10	559 (n.c-f)
Proportion of 1s (p_j)	0.267	0.133	0.467	0.133	0.333	0.267($\bar{P}$)

Using the proportions in Table 2 in Equ 18 we have the chi-square test statistic for the null hypothesis of equ 5 that candidates are equally likely to earn the highest scores under each of the five judges as

$$\text{i}^{2} = \frac{(15)(0.267)^2(0.133)^2 + (0.467)^2 + (0.133)^2 + (0.333)^2 - 950(0.267)^2)}{(0.267)(0.733)}$$

$$= \frac{(15)(0.436 - 0.355)}{0.196} = \frac{15(0.081)}{0.196} = \frac{1.215}{0.196} = 6.199(P - value = 0.1463)$$

Which with 4 degrees of freedom is not statistically significant leading to the acceptance of the null hypothesis. Note that if in fact the null hypothesis had been rejected and further research interest were in determining which of the judges may have importantly contributed to the rejection of H_0,then one would have proceeded to temporarily omit the scores by judge 3 which is here seen to make the large contribution to the calculated total chi-square value of 6.199.We now here use Cochran Q test to reanalyze the data of table 2 to enable comparison of the proposed method with an existing method that may alternatively be used. The usual Cochran Q test statistics is

$$Q = \frac{(c-1)\left(\sum_{j=1}^{c} T_j^{\,2} - \left(\sum_{j=1}^{c} T_j\right)^2 \Big/ c\right)}{\sum_{i=1}^{n} B_i - \sum_{j=1}^{n} B_i^2 \Big/ c} \qquad 20$$

Which has chi-square distribution with c-1 degrees of freedom. Using the summary values of f_j and B_i in Table 2,we have

$$Q = \frac{(5-1)\left((4^2 + 2^2 + 7^2 + 2^2 + 5^2 - (20)^2 \big/ 5\right)}{20 - \left(1^2 + 1^2 + \ldots + 1^2\right)\big/ 5}$$

$$= \frac{4(98-80)}{20-6} = \frac{4(18)}{14} = \frac{72}{14} = 5.143(P - value = 0.1780)$$

Which with 4 degrees of freedom is also not statistically significant again leading to the acceptance of the null hypothesis. Although the Cochran Q test and the proposed method here lead to the same conclusion. The relative sizes of the corresponding chi-square values and the attained significance levels indicate that the Cochran Q test at least for the present example is likely to lead to a acceptance of a false null hypothesis (Type II Error)more often, and hence is likely to be less powerful than the proposed method.

SUMMARYAND CONCLUSION

We have in this paper proposed and developed a non-parametric statistical method for the analysis of maximum (or minimum) scores by subjects exposed to a battery of tests over time or space. The proposed method may be used for analyzing data measured on as low as the ordinal scale that are not necessarily continuous or numeric. A chi-square test statistic is developed to test the null hypothesis that subjects are on the average equally likely to perform or earn the highest (best, largest) scores under various treatment levels, time period, conditions or locations. In the event that the null hypothesis is rejected, the proposed method may also be used to identify the treatment level or treatment levels that may have accounted for the rejection of the null hypothesis. The proposed method is illustrated with some sample data and shown to compare favorably with the Cochran Q test.

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