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Research Article

Construction of an Exact Solution of Time-Dependent Ginzburg-Landau Equations by Standard Integral for Front Propagation in Superconductors

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Abstract

We present a new approach for front propagation at which the interface moves from a superconducting to normal region in a superconducting sample. Using the time dependent Ginzburg-Landau (TDGL) equations we study the interface propagation by constructing the exact solution for order parameter.

Keywords: Front propagation in superconductor, Time-dependent Ginzburg-Landau

Running title: Construction of an exact solution of time-dependent Ginzburg-Landau equations.

Introduction

The theory of propagating fronts in systems with a continuous order parameter is considered, where a stable state invades an unstable state. It is found that if the system is suddenly made unstable, the subsequent dynamics is characterized by the propagation of fronts [1]. Such front arise in models of fluid dynamics, population growth, the propagation of superconducting front into a normal metal [2,3].In recent years the phenomena of magnetic field penetration or its expulsion from superconducting sample is a great attraction to different research groups .Fisher[4] and Kolmogorov et al.[5] worked first with fronts in nonlinear diffusion type equation. The prototype of such equations is the parabolic reaction diffusion equation

$$u_t = u_{xx} + F(u) \text{ and } F(0)=F(1)=0 \text{ and } F>0$$

The non linear term $F(u)$ is such that there exist two steady states, one stable and one unstable. For positive reaction terms $F(u)>0$, sufficiently localized initial conditions evolve into traveling wave fronts of the form $u(x,t)=U(x-vt)$ connecting the stable state $u=1$ to the unstable state $u=0$ [6]. *X= x – vt* is a traveling coordinate, moving to the right (for *v* >0) with velocity *v.* This coordinate is suitable for examining traveling front.

For the evolutionary description of the system between two homogeneous equilibrium states, it is necessary to embed a superconducting sample in a stationary applied magnetic field equal to the critical field H_C. After removing the magnetic field rapidly, the planar superconducting-normal interface is dynamically unstable and propagates toward the normal phase, hereby expelling any trapped magnetic flux. As a result the sample is in Meissner state [7].

In this paper we have presented the detailed procedure for the construction of an exact solution of the time-dependent Ginzburg-Landau (TDGL) equations [8] for superconducting fronts. The starting points are the TDGL equations.

Materials and Method

Di Bartolo et al. [2] are interested in calculating traveling wave solutions for the propagation of one dimensional front. In dimensionless units [8] the one dimensional TDGL equations in this case reduce to

$$\partial_t f = \frac{1}{\kappa^2}\partial_x^{\,2} f - q^2 f + f - f^3 \qquad (1)$$

$$\bar{\sigma}\,\partial_t q = \partial_x^2 q - f^2 q \qquad (2)$$

Here, the quantity f is the magnitude of the superconducting order parameter ψ, q is the gauge-invariant vector potential connected to the magnetic field as $h=\frac{\partial q}{\partial x}$, $\bar{\sigma}$ is the dimensionless normal state conductivity (the ratio of the order parameter diffusion constant to the magnetic field diffusion constant) and κ is the Ginzburg-Landau parameter.

Using the time-dependent Ginzburg-Landau (TDGL) equations, we desire a more general approach to the problem of superconducting front. The TDGL equations are

$$\frac{1}{\kappa^2}F'' + vF' - Q^2F + F - F^3 = 0 \qquad (3)$$

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$$Q'' + \bar{\sigma} v Q' - F^2 Q = 0 \quad (4)$$

In which magnitude of the superconducting order parameter *F* and vector potential *Q* depend on both space and time. The TDGL equations have the traveling wave solution of the form

f(x,t)=F(x - vt) = F(X) and q(x,t)= Q(x - vt) = Q(X) , where v is the propagating speed of a front in superconductor.

We assume that the vector potential is a function of magnitude of the superconducting order parameter. This is always possible,
i.e. *Q = Q(F)* (5)

Then the derivative will be $Q_F = \frac{dQ}{dF}$ (6)

Now, $F' = \frac{dF}{dX}$ and $Q' = \frac{dQ}{dX}$ so that

$$Q' == Q_F F' \quad (7)$$

$$Q'' = Q_{FF} F'^2 + Q_F F'' \quad (8)$$

We substitute from eq. (7) and eq. (8) into eq. (4) to get the following equation

$$F'' + \left(Q_{FF}/Q_F\right)F'^2 + \bar{\sigma} v F' - F^2\left(Q/Q_F\right) = 0 \quad (9)$$

Equation (3) can be written as

$$F'' + \kappa^2 v F' + \kappa^2\left(1 - Q^2\right)F - \kappa^2 F^3 = 0 \quad (10)$$

By subtracting eq. (10) from eq. (9) we obtain a single equation of the following form:

$$\left(Q_{FF}/Q_F\right)F'^2 + v\left(\bar{\sigma} - \kappa^2\right)F' + \kappa^2\left(Q^2 - 1\right)F + \left(\kappa^2 F - Q/Q_F\right)F^2 = 0 \quad (11)$$

This is a non- linear differential equation. Now let us choose

$$Q_{FF} = 0 \quad (12a)$$

This is possible if the vector potential is related to order parameter by arbitrary constants a and b like as

Q = aF + b (12b)

This gives $Q_F = a$ (12c)

and $\frac{Q}{Q_F} = F + \frac{b}{a}$ (12d)

To evaluate eq. (11) substitution from eq. (12a, b and d) implies

$$v\left(\bar{\sigma} - \kappa^2\right)\frac{dF}{dX} = F\left\{\left(1 - \kappa^2 a^2 - \kappa^2\right)F^2 + \left(\frac{b}{a} - 2ab\kappa^2\right)F + \left(\kappa^2 - \kappa^2 b^2\right)\right\} \quad (13)$$

Let us define three parameters in terms of front speed, G-L parameter and dimensionless normal state conductivity as

$$p_2 = \left(1 - \kappa^2 a^2 - \kappa^2\right) \Big/ v\left(\bar{\sigma} - \kappa^2\right) \quad (14a)$$

$$p_1 = b/a - 2ab\kappa^2 \Big/ v\left(\bar{\sigma} - \kappa^2\right) \quad (14b)$$

$$p_0 = \kappa^2\left(1 - b^2\right) \Big/ v\left(\bar{\sigma} - \kappa^2\right) \quad (14c)$$

Now, we use these parameters in eq. (13) and integrating

$$\int dX = \int \frac{dF}{F\left(p_2 F^2 + p_1 F + p_0\right)} \quad (15)$$

Equation (15) is a standard integral which can be evaluated. One finds that the superconducting front is the solution of eq. (15).

Result and Discussion

The expression for the integration is given by

$$\int \frac{dF}{F\left(p_2 F^2 + p_1 F + p_0\right)} = X + X_0 \quad (16)$$

Where X_0 is integration constant.

The integral (16) is quite complicated as $p_2 F^2 + p_1 F + p_0$ is the quadratic trinomial and powers of F. We show how to evaluate the expression on the left hand side of (16).

Let $I = \int \frac{dF}{F\left(p_2 F^2 + p_1 F + p_0\right)}$ (17)

Decomposing into partial fractions

$$I = \int \frac{\alpha}{F} dF + \int \frac{\beta F + \gamma}{p_2 F^2 + p_1 F + p_0} dF$$

Where $\alpha = \frac{1}{p_0}$ $\beta = -\frac{p_2}{p_0}$ and $\gamma = -\frac{p_1}{p_0}$

$$\therefore I = \frac{1}{p_0}\log F - \frac{1}{2p_0}\log\left(p_2 F^2 + p_1 F + p_0\right) - \frac{p_1}{2p_0} I' \quad (18)$$

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With $I' = \int \frac{dF}{p_2F^2 + p_1F + p_0} = \frac{1}{p_2}\int \frac{dF}{\left(F + \frac{p_1}{2p_2}\right)^2 + \frac{4p_2p_0 - p_1^2}{4p_2^2}}$ (19)

Write $4p_2p_0 - p_1^2 = \Delta$

Then I' leads to

$$I' = \frac{1}{p_2}\int \frac{dF}{\left(F + \frac{p_1}{2p_2}\right)^2 + \frac{\Delta}{4p_2^2}} \tag{20}$$

We get three cases which are as follows:

Case 1: $\Delta > 0$, Case 2: $\Delta < 0$ and Case 3: $\Delta = 0$.

Now we will discuss three cases.

Case 1: $\Delta > 0$: The only positive solution is one assuming Δ to be positive. For mathematical simplification we take $\frac{\Delta}{4p_2^2} = \zeta^2$ then eq. (20) takes the form

$$I_1' = \frac{1}{p_2}\int \frac{dF}{\left(F + \frac{p_1}{2p_2}\right)^2 + \zeta^2} \tag{21}$$

Let us write $F + \frac{p_1}{2p_2} = y = \zeta \tan\theta$ (22)

$\therefore\ dF = dy = \zeta \sec^2\theta d\theta$ Using this, eq. (21) implies

$$I_1' = \frac{1}{p_2}\int \frac{\zeta \sec^2\theta d\theta}{\zeta^2(\tan^2\theta + 1)} = \frac{1}{p_2\zeta}\int d\theta = \frac{1}{p_2\zeta}\theta \tag{23}$$

From (22)

$$\theta = \tan^{-1}\left(\frac{F}{\zeta} + \frac{p_1}{2p_2\zeta}\right)$$

Therefore, $I_1' = \frac{1}{p_2\zeta}\tan^{-1}\left(\frac{F}{\zeta} + \frac{p_1}{2p_2\zeta}\right)$ (24)

Hence for $\Delta > 0$ the full solution of eq. (15) takes the form together with eq. (18) and (24) as

$$X + X_0 = \frac{1}{p_0}\log F - \frac{1}{2p_0}\log\left(p_2F^2 + p_1F + p_0\right) - \frac{p_1}{2p_0p_2\zeta}\tan^{-1}\left(\frac{F}{\zeta} + \frac{p_1}{2p_2\zeta}\right) \tag{25}$$

This is an implicit solution.

Case 2: $\Delta < 0$; i.e. Δ is negative which means that $p_1^2 > 4p_2p_0$. In this case we do not have a positive value so we take

$$\eta^2 = -\frac{\Delta}{4p_2^2} \tag{26}$$

Then eq. (20) takes the form

$$I_2' = \frac{1}{p_2}\int \frac{dF}{\left(F + \frac{p_1}{2p_2}\right)^2 - \eta^2} \tag{27}$$

Now let us introduce $F + \frac{p_1}{2p_2} = y = \eta \tanh\vartheta$ (28)

Hence we have $dF = dy = \eta d(\tanh\vartheta) = \eta(1 - \tanh^2\vartheta)d\vartheta$

$$\therefore\ I_2' = \frac{1}{p_2}\int \frac{\eta(1 - \tanh^2\vartheta)d\vartheta}{\eta^2\tanh^2\vartheta - \eta^2} = -\frac{1}{p_2\eta}\vartheta \tag{29}$$

From eq. (28) we have $\vartheta = \tanh^{-1}\left(\frac{F}{\eta} + \frac{p_1}{2p_2\eta}\right)$

Therefore $I_2' = -\frac{1}{p_2\eta}\tanh^{-1}\left(\frac{F}{\eta} + \frac{p_1}{2p_2\eta}\right)$ (30)

In analogy with $\tanh^{-1}x = \frac{1}{2}\log\frac{1+x}{1-x}$ I_2' can be written as

$$I_2' = -\frac{1}{p_2\eta}\frac{1}{2}\log\frac{1 + \frac{F}{\eta} + \frac{p_1}{2p_2\eta}}{1 - \frac{F}{\eta} - \frac{p_1}{2p_2\eta}}$$

We shall see from (26) that $\eta = \sqrt{-\Delta}/2p_2$ which can transform I_2' as follows:

$$I_2' = -\frac{1}{p_2\sqrt{-\Delta}/2p_2}\frac{1}{2}\log\frac{2p_2\eta + 2p_2F + p_1}{2p_2\eta - 2p_2F - p_1}$$

$$= -\frac{1}{\sqrt{-\Delta}}\log\frac{\sqrt{-\Delta} + (p_1 + 2p_2F)}{\sqrt{-\Delta} - (p_1 + 2p_2F)} \quad (31)$$

Hence for $\Delta < 0$ the corresponding complete solution of eq. (15) for order parameter F is as follows (by using (18) and (31))

$$X + X_0 = \frac{1}{p_0}\log F - \frac{1}{2p_0}\log\left(p_2F^2 + p_1F + p_0\right) + \frac{p_1}{2p_0\sqrt{-\Delta}}\log\frac{\sqrt{-\Delta} + (p_1 + 2p_2F)}{\sqrt{-\Delta} - (p_1 + 2p_2F)} \quad (32)$$

This is a complicated solution.

Case 3: $\Delta = 0$; i.e. $4p_2p_0 = p_1^2$

For this case eq. (20) yields

$$I_3' = \frac{1}{p_2}\int\frac{dF}{\left(F + \frac{p_1}{2p_2}\right)^2}$$

$$= -\frac{1}{p_2}\frac{1}{\left(F + \frac{p_1}{2p_2}\right)} = -\frac{2}{p_1 + 2p_2F} \quad (33)$$

Therefore the complete solution of eq. (15) for $\Delta = 0$ will be given by (by using eq.(18) and eq.(33))

$$\mathrm{X} + \mathrm{X}_0 = \frac{1}{\mathrm{p}_0}\log\mathrm{F} - \frac{1}{2\mathrm{p}_0}\log\left(\mathrm{p}_2\mathrm{F}^2 + \mathrm{p}_1\mathrm{F} + \mathrm{p}_0\right) + \frac{\mathrm{p}_1}{\mathrm{p}_0(\mathrm{p}_1 + 2\mathrm{p}_2\mathrm{F})} \quad (34)$$

Conclusion

We have studied the front propagation into unstable state. The state F=0 is unstable and F=± 1 are stable states. Our solution is presented in a standard integral form (15) where we assumed that the vector potential *Q* is a function of the magnitude of superconducting order parameter *F*. Here we have obtained three types of solutions (25), (32) and (34) for three parametric cases. These solutions are very complicated. Extension of these results will be our future work which may offer new physical insights of front propagation in superconductivity at the superconducting-normal interface. There may possibly be applications for specific values of the parameters occurring in the solutions for the laboratory experiments examining these important phenomena. Some simple interesting situations may obtained by setting one or more parameters equal to zero.

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