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Some Statistical Analysis for Continuous – Time Stationary Processes with Missed Data

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Abstract: In this paper, the statistical analysis for continuous-time stationary processes with missed data is presented. The statistics of this process are constructed. The asymptotic properties (first order, second order and k – th order cumulant) for this process are investigated. The asymptotic distribution for an estimate of the spectral density function is established

Keywords: Time series, asymptotic distribution, missed data, continuous time stationary process, spectral density function, normal distribution.

Introduction

Let $X^r(t) = \{X_a(t)\ ,\ a = \overline{1\,,r},\ t \in \boldsymbol{R}\}$ be an r-dimensional continuous time stationary process with mean m_a, $a = \overline{1\,,r}$, $R_{aa}(\tau)\ ,\tau \in \boldsymbol{R}$ the continuous covariance function which is defined by

$$\boldsymbol{R}_{aa}(\tau) = E\{X_a(t+\tau)\, X_a(t)\} = \int_R f_{aa}(\lambda)\, exp(i\,\lambda\,\tau) d\lambda\,; t\,,\tau \in R \qquad \textbf{(1.1)}$$

and the spectral density function $f_{aa}(\lambda)$, $\lambda \in \boldsymbol{R}, \boldsymbol{a} = \overline{1\,,r}$ which is given by

$$f_{aa}(\lambda) = (2\,\pi)^{-1} \int_{\boldsymbol{R}} R_{aa}(\tau)\, exp(-\,i\,\lambda\,\tau)\, d\tau, \lambda \in \boldsymbol{R}\,. \qquad \textbf{(1.2)}$$

Consider the statistic:

$$\widehat{\boldsymbol{m}_a} = \frac{1}{T}\int_0^T X_a(t)\, dt, \qquad \textbf{(1.3)}$$

Which is constructed by T, $T = [\,1\,,\infty)$, $X_a(t)$ is a sequence of observations on $X^r(t)$. Let

$$Y_a(t) = X_a(t)\, d_a(t), \qquad \textbf{(1.4)}$$

is irregularly data on the stationary process $Y^r(t), a = \overline{1\,,r}$ and $d_a(t)$ is Bernoulli sequence of random variables, which satisfies

$$\boldsymbol{d_a}(t) = \begin{cases} 1 & if\ X_a(t)\ observed \\ 0 & otherwise \end{cases}$$

(1.5)

Since $P\{d_a(t) = 1\} = p_a > 0, P\{d_a(t) = 0\} = q_a,\ p_a + q_a = 1, a = \overline{1\,,r}$ and **(1.6)**

Several authors as e.g. Bloomfield (1970), Brillinger (1970), Marshal (1980) studied the asymptotic time series with missing observations. Ghazal (1999), Ghazal and Farag (1998a) and (1998b) studied the asymptotic time stationary processes with classical methods.

The paper is organized as follows: Estimation of the expectation and its asymptotic distribution is determined in Section 2. Section 3 contains the asymptotic distribution for continuous – time processes with missed data.

2. Estimation of the expectation and its asymptotic distribution

We mention some results, which will be used to prove some theorems.

Theorem 2.1 For estimation $\widehat{m_a}$, which is defined by equation (1.3), then we has

$$E\,\widehat{m_a} = m_a$$

(2.1)

$$T\, cov\{\widehat{m_a}\,,\widehat{m_b}\} = 2\pi \int_{\boldsymbol{R}} f_{aa}(x)\, \Phi_T(x)\, dx, \qquad \textbf{(2.2)}$$

Where

$$\Phi_T(x) = (2\,\pi\,T)^{-1}\, \frac{sin^2 x\,T\,/2}{(x/2)^2},$$

(2.3)

$T\,D\widehat{m_a} = 2\pi \int_{\boldsymbol{R}} f_{aa}(x)\, \Phi_T(x)\, dx,$
(2.4)

for all $x \in \boldsymbol{R}$ and $a\,, b = \overline{1\,, r}$.

Proof: Equation (2.1) comes directly by using formula (1.3). From equation (1.1), we get

$cov\,(\widehat{m_a}\,, \widehat{m_b})$

$$= T^{-2} \int_{\boldsymbol{R}} f_{ab}(x) \left\{ \int_0^T exp(i\,x\,t_1)\, dt_1 \int_0^T exp(-\,i\,x\,t_2)\, dt_2 \right\} dx$$

Since

$\int_0^T exp(i\,x\,t)\, dt = \frac{\sin x\, T/2}{x/2}\, exp(i\,x\,T/2),$

and by using formula (2.3), then equation (2.2) is obtained. Equation (2.4) comes directly by putting $a = b$ in equation (2.2). Now, the proof of the theorem is completed.

Now, theorem (2.2) below shall study the properties of the kernel $\Psi_T(x)$ on $\boldsymbol{R}$, $x \in \boldsymbol{R}$ which is defined by:

$\Psi_T(x) = \; (2\,\pi\,T)^{-\mathbf{1}} \left| \int_0^T exp(-\,i\,x\,t)\, dt \right|, T = 1, 2,$ **(2.5)**

Theorem 2.2 For any $x \in \boldsymbol{R}$, the function $\Psi_T(x)$ is the kernel on $\boldsymbol{R}$ that satisfy the following properties

(1) $\int_{-\infty}^{\infty} \Psi_T(x)\, dx = 1,\; x \,\epsilon\, \boldsymbol{R}.$ **(2.6)**

(2) $\lim_{T\to\infty} \int_{-\infty}^{-\delta} \Psi_T(x)dx = \lim_{T\to\infty} \int_{\delta}^{\infty} \Psi_T(x)dx = 0,$ for any $\delta >, x \in \boldsymbol{R},$ **(2.7)**

(3) $\lim_{T\to\infty} \int_{-\delta}^{\delta} \Psi_T(x)dx = \, 1,$ for all $\delta > 0, x \; \in \boldsymbol{R}.$

(2.8)

Proof: The proof is omitted.

Lemma 2.1 If the function $g(x)$, $x \in \boldsymbol{R}$ is bounded and continuous at a point $x = 0$ and the function $\Psi_T(x)$, $x \in \boldsymbol{R}$ satisfies the properties of Theorem 2.2, then

$\lim_{T\to\infty} \int_{-\infty}^{\infty} g(x)\, \Psi_T(x)dx = \, g(0).$ **(2.9)**

Proof: By using formula (2.6), we get

$$\left| \int_{-\infty}^{\infty} g(x)\Psi_T(x)dx - g(0) \right| \le \int_{-\infty}^{-\delta} |\,g(x) - g(0)|\,|\Psi_T(x)|dx + \int_{-\delta}^{\delta} |\,g(x) - g(0)|\,|\Psi_T(x)|dx +$$

$$+ \int_{\delta}^{\infty} |\,g(x) - g(0)|\,|\Psi_T(x)|dx = A_1 + A_2 + A_3.$$

Now, from continuity of $g(x)$ at $x = 0$, we get

$A_2 \le \varepsilon \int_{-\infty}^{\infty} |\Psi_T(x)|dx$. Then, $A_2 \le \varepsilon$

Now, A_2 is very small according any ε is very small. Then $A_2 = 0$. Suppose that $g(x), x \in \boldsymbol{R}$ is bounded by constant M, then according to formula (2.7),

$A_1 \le 2M \int_{-\infty}^{-\delta} |\Psi_T(x)|dx \xrightarrow[T\to\infty]{} 0.$ Similarly, $A_3 \xrightarrow[T\to\infty]{} 0.$

Therefore, $\int_{-\infty}^{\infty} g(x)\Psi_T(x)dx - g(0) \xrightarrow[T\to\infty]{} 0,$ this completes the proof of the theorem.

Theorem 2.3 If the spectral function $f_{ab}(x), x \in \boldsymbol{R}$, $a, b = \overline{1, r}$ is bounded and continuous at a point $x = 0$, then

(1) $\lim_{T\to\infty} T\, cov\{\hat{m}_a\,, \hat{m}_b\} = 2\pi\, f_{ab}(0),\; a\,, b = \overline{1, r}$ **(2.10)**

(2) $\lim_{T\to\infty} T\, D\hat{m}_a = 2\pi\, f_{aa}(0),\; a = \overline{1, r}$ **(2.11)**

Proof: The proof can be easily obtained by using lemma 2.1.

Theorem 2.4 Let $X^r(t) = \left\{ X_a(t), t \in \boldsymbol{R}, a = \overline{1, r} \right\}$, be r – dimensional continuous stationary process with mean zero. Then the statistic

$$\Theta_a(T) = \frac{1}{\sqrt{T}} \int_0^T X_a(t)dt, \quad \mathbf{(2.12)}$$

is asymptotically normal distribution with mean zero and dispersion given by

$$\lim_{T\longrightarrow\infty} D\Theta_a(T) = 2\pi\, f_{aa}(0),\ a=\overline{1,r}.$$

Proof: We begin by noting that $E\{\Theta(T)\}=0$ according to the condition that $E\,X_a(t)=0$, $a=\overline{1,r}$. Next we note from formula (1.2) that

$$cov\{\Theta_a(T),\Theta_b(T)\} = \frac{1}{T}\int_{-\infty}^{\infty} f_{ab}(x)\int_0^T\int_0^T exp\{i\,x(t_1-t_2)\}\,dt_1\,dt_2\,dx$$

=

=

$$2\pi\int_{-\infty}^{\infty} f_{ab}(x)\left\{(2\pi T)^{-1}\int_0^T exp(i\,x t_1)dt_1\int_0^T exp(-i\,x t_2)dt_2\right\}dx$$

.

By using equation (2.5), then

$$cov\{\Theta_a(T),\Theta_b(T)\} = 2\pi\int_{-\infty}^{\infty} f_{ab}(x)\,\Psi_T(x)dx.$$

Taking the limits on both sides and then by using lemma 2.1 and then putting $a=b$, we get

$$\lim_{T\longrightarrow\infty} D\{\Theta_a(T)\} = 2\pi\, f_{aa}(0),\ a=\overline{1,r}.$$

Finally,

$$cum\{\Theta_{a_1}(T),\ldots,\Theta_{a_k}(T)\} = T^{-k/2}\int_0^T\ldots\int_0^T cum\{X_{a_1}(t),\ldots,X_{a_k}(t)\}dt_1\ldots dt_k.$$

$$cum\{\Theta_{a_1}(T),\ldots,\Theta_{a_k}(T)\} = T^{-k/2}\int_0^T\ldots\int_0^T c_{a_1\ldots a_k}(t_1-t_k,\ldots,t_{k-1}-t_k)dt_1\ldots dt_k$$

Putting $t_1-t_k=u_1,\ldots,t_{k-1}-t_k=u_{k-1}$, then

$$\left|cum\{\Theta_{a_1}(T),\ldots,\Theta_{a_k}(T)\}\right| \le T^{-k/2}\int_0^T\int_{-T}^T\ldots\int_{-T}^T\left|c_{a_1\ldots a_k}(u_1,\ldots,u_{k-1})\right|du_1\ldots du_{k-1}\,dt.$$

Suppose

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\left|c_{a_1\ldots a_k}(u_1,\ldots,u_{k-1})\right|du_1\ldots du_{k-1}<\infty,\ k=2,3,\ldots$$

Then

$$cum\{\Theta_{a_1}(T),\ldots,\Theta_{a_k}(T)\} = O\left(T^{1-k/2}\right)\xrightarrow[T\longrightarrow\infty]{}0,$$

for all $k>2$.

Then $\Theta_a(T)=\frac{1}{\sqrt{T}}\int_0^T X_a(t)\,dt$ is asymptotically normal distribution with mean zero and dispersion given by $2\pi\, f_{aa}(0)$, which completes the proof.

3. Asymptotic distribution of stationary process with missed observations

Now, we will study the moments and the asymptotic distribution for $Y_a(t),\, t\in\boldsymbol{R}$ which is defined by equation (1.4).

Lemma 3.1 If the process $X_a(t)$ and $Y_a(t), a=\overline{1,r}$ corresponding by equation (1.4) where a sequence $d_a(t), t\in\boldsymbol{R}, a=\overline{1,r}$ satisfies equations (1.5) and (1.6), m'_a is the mean, $R'_{aa}(\tau),\ \tau\in\boldsymbol{R}$ is the covariance function and $f'_{aa}(\lambda),\ \lambda\in\boldsymbol{R},\ a=\overline{1,r}$ is the spectral density function on the processes $Y_a(t)$, $t\in\boldsymbol{R}$, $a=\overline{1,r}$, then

(1) $\quad m'_a = m_a\ p_a,$ **(3.1)**

(2)
$$R'_{aa}(\tau)=\begin{cases} p_a R_{aa}(0)+m_a^2\ p_a\,q_a & if\ \tau=0\\ p_a^2\,R_{aa}(\tau) & if\ \tau\neq 0\end{cases}\qquad \textbf{(3.2)}$$

(3)
$$f'_{aa}(\lambda) = p_a^2\ f_{aa}(\lambda)+(2\pi)^{-1}\,p_a\,q_a\left[m_a^2+R_{aa}(0)\right]\qquad \textbf{(3.3)}$$

Proof: Formula (3.1) can be deduced from equation (1.4) and properties of $d_a(t)$, $a=\overline{1,r}$. To prove formula (3.2), we get

$$R'_{aa}(\tau)=E\,X_a(t+\tau)\,X_a(t)E\,d_a(t+\tau)\,d_a(t)-E\,X_a(t+\tau)\,E\,X_a(t)E\,d_a(t+\tau)\,E\,d_a(t).$$

Since

$$E\,X_a(t+\tau)\,X_b(t)=R_{ab}(\tau)+m_a^2,\qquad \textbf{(3.4)}$$

Then

$$R'_{aa}(\tau)=\begin{cases} E\,X_a(t)X_a(t)E\,d_a(t)d_a(t)-E\,X_a(t)E\,X_a(t)\,E\,d_a(t)\,E\,d_a(t), & if\ \tau=0\\ E\,X_a(t+\tau)X_a(t)E\,d_a(t+\tau)\,d_a(t)-m_a^2\,p_a^2 & if\ \tau\neq 0\end{cases}$$

By using equation (3.4) and properties of $d_a(t),\ a=\overline{1,r}$, we have

$$R'_{aa}(\tau)=\begin{cases}\left(R_{aa}(0)+m_a^2\right)p_a-m_a^2\,p_a^2\,, & \tau=0\\ \left(R_{aa}(\tau)+m_a^2\right)p_a^2-m_a^2\,p_a^2\,, & \tau\neq 0\end{cases}$$

Then formula (3.2) is obtained. Finally, from the definition of spectral density and equation (3.2), we get

$$f'_{aa}(\lambda)=(2\pi)^{-1}\left\{\left[\,p_a\,R_{aa}(0)+m_a^2\,p_a\,q_a\,\right]+p_a^2\,R_{aa}(\tau)exp(-i\,\lambda\,\tau)\,\Big|_{\tau=0}\right.$$

$$+\qquad \left. p_a^2\int_{-\infty}^{\infty}R_{aa}(\tau)\,exp(-i\,\lambda\,\tau)d\tau\right\}=$$

$$(2\pi)^{-1}\left\{\left[\,p_a\,R_{aa}(0)+m_a^2\,p_a\,q_a\,\right]+p_a^2\,R_{aa}(0)+2\pi\,p_a^2\,f_{aa}(\lambda)\right\}$$

$$=$$

$$p_a^2\,f_{aa}(\lambda)+(2\pi)^{-1}\,p_a\,q_a\,R_{aa}(0)+(2\,\pi)^{-1}\,m_a^2\,p_a\,q_a$$

.

Then equation (3.3) is obtained and then the proof is completed.

Theorem 3.1 Let

$$\overline{m_a}=\frac{1}{T\,p_a}\int_0^T Y_a(t)\,dt\,, \qquad \textbf{(3.5)}$$

Then

$$E\,\overline{m_a}=m_a\,, \qquad \textbf{(3.6)}$$

$$D\overline{m_a}=\frac{1}{T\,p_a}\left[\int_{-\infty}^{\infty}f_{aa}(y)\,dy+m_a^2\,q_a\right]+\frac{2\pi}{T}\int_{-\infty}^{\infty}f_{aa}(y)\,\Phi_T(y)\,dy\,, \qquad \textbf{(3.7)}$$

where $\Phi_T(y)$ is defined by equation (2.3).

Proof: Formula (3.5) comes directly from equation (1.4). To prove formula (3.6), then by using equation (3.5) and the definition of the dispersion to get

$$D\overline{m_a}=\frac{1}{T^2\,p_a^2}\int_0^T\int_0^T\left\{E\,Y_a(t_1)\,Y_a(t_2)-E\,Y_a(t_1)\,E\,Y_a(t_2)\right\}dt_1$$

.

By using equations (1.4) and (3.2), we get

$$D\overline{m_a}=\frac{1}{T^2}\int_0^T\int_0^T R_{aa}(t_1-t_2)\,dt_1\,dt_2+\frac{R_{aa}(0)+m_a^2\,q_a}{T\,p_a}$$

By using equation (1.1), then formula (3.7) is obtained and the proof is completed.

Now, we will study the limiting distribution for the statistic defined by (3.5).

Corollary 3.1 Let $\overline{m_a}$ be defined by equation (3.5). If the spectral density $f_{aa}(\lambda)$ is continuous at a point $\lambda=0$ and bounded on $\boldsymbol{R}$ then

$$\lim_{T\to\infty}\,T\,D\overline{m_a}=2\pi\,f_{aa}(0)+\frac{1}{p_a}\left[m_a^2\,q_a+\int_{-\infty}^{\infty}f_{aa}(y)\,dy\right] \qquad \textbf{(3.8)}$$

Proof: The proof can be obtained directly from equation (3.7) as $T\longrightarrow\infty$.

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